A Generalized Recursive Algorithm for Wave-Scattering Solutions in Two Dimensions

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Abstract—A generalized recursive algorithm valid for both the E_z and H_z wave scattering of densely packed scatterers in two dimensions is derived. This is unlike previously derived recursive algorithms which have been found to be valid only for E_z polarized waves [1]–[7]. In this generalized recursive algorithm, a scatterer is first divided into N subscatterers. The n-subscatterer solution is then used to solve the $(n + n')$-subscatterer solution. The computational complexity of such an algorithm is found to be of $O(N^2)$ in two dimensions, and meanwhile, providing a solution valid for all angles of incidence. This is better than the method of moments with Gaussian elimination which has an $O(N^3)$ complexity.

I. INTRODUCTION

RECURSIVE algorithms for calculating multiple scattering of many scatterers have recently been developed [1]–[7]. The recursive algorithm calculates the scattering solutions considerably faster than the conventional method of moments with Gaussian elimination [6] because of the reduced computational complexity of the recursive algorithm, which is of $O(N^2)$ where N is the number of unknowns used to model the inhomogeneous scatterer. Moreover, unlike the conjugate gradient method [9]–[11], the recursive algorithm provides a full scattering solution valid for all angles of incidence.

In the aforementioned algorithm, an inhomogeneous scatterer is first divided into N subscatterers. Then the solution to the N subscatterer problem is sought recursively, namely, the n-subscatterer solution is used to derive the solution of $(n + 1)$ subscatterers. In order to facilitate the use of the addition theorem, the subscatterers are ordered so that they are at increasing distances from the origin [3]–[5].
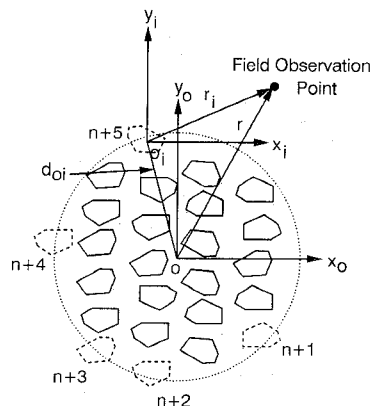
Fig. 1. In the recursive algorithm, n-subscatterer solution is used to find the $(n + n')$-subscatterer solution, where n' is the number of subscatterers equidistant from the origin. In this figure, $n' = 5$.

Unfortunately, when the scatterers are tightly packed together, some of the subscatterers are almost equidistant from the origin [see Fig. 1], violating the use of the addition theorem. For an E_z polarized wave when μ is a constant, this minor violation of the addition theorem is not serious as the scattered field from each subscatterer is predominantly monopole. In two dimensions, the monopole field is only logarithmically singular: a weak singularity which does not pose a serious problem when the addition theorem is violated. However, the previously described algorithm [3]–[5] does not work when it is used to calculate the scattering of an H_z polarized wave by an inhomogeneous cylinder modeled by a cluster of tightly packed subscatterers. The reason is that the scattered field from each subscatterer is predominantly dipolar, which is more singular than a monopolar field. Hence, the infraction of the addition theorem is not tolerable in this case.

As a remedy, we present a generalization of the previous algorithms so that the infraction of the addition theorem does not occur. In this generalized algorithm, n' subscatterers, which are nearly equidistant from the origin, are added at once to the previous group of n subscatterers at each recursion [8]. Hence, the n-subscatterer solution is used to find the $(n + n')$-subscatterer solution. This generalized algorithm is found to work well when it is used to construct scattering solutions of H_z polarized waves from an inhomogeneous cylinder. As a demonstration, we shall also use this algorithm to calculate the scattering of an H_z polarized wave from an array of strips.

II. THE GENERALIZED RECURSIVE ALGORITHM

Consider the case where an inhomogeneous scatterer is divided into N subscatterers. To find a recursive solution, one assumes that the n-subscatterer solution is known, with the total field expressible as

$$\phi(r) = \Re g\psi^t(k_0, r_0) \cdot a + \psi^t(k_0, r_0) \cdot \overline{\tau}_{(n)} \cdot a. \quad (1)$$

In the above, $\psi(k_0, r_0)$ is a column vector containing the cylindrical harmonics in two dimensions and spherical harmonics in three dimensions, and k_0 is the wavenumber of the homogeneous medium in which the scatterer is residing. In two dimensions, assuming $e^{-i\omega t}$ time dependence, the mth element of $\psi(k_0, r_0)$ is

$$[\psi(k_0, r_0)]_m = H_m^{(1)}(k_0\rho_0)e^{im\phi_0}, \quad m = -L, \cdots, L, \quad (1a)$$

where ρ_0 and ϕ_0 are coordinates with respect to global coordinates with origin 0 as shown in Fig. 1, and $H_m^{(1)}(x)$ is an m-order Hankel fucntion of the first kind. "$\Re g$" stands for "regular part." Hence

$$[\Re g\psi(k_0, r_0)]_m = J_m(k_0\rho_0)e^{im\phi_0}, \quad m = -L, \cdots, L, \quad (1b)$$

where $J_m(x)$ is an m-order Bessel function.

The first term of (1) denotes the incident wave expanded in terms of Bessel wave functions (or standing waves), whose amplitudes are contained in the column vector a. The second term of (1) denotes the scattered wave functions off the n subscatterers [since (1a) represents outgoing wave functions], whose amplitudes are related to the amplitudes of the incident wave functions via the aggregate T matrix $\overline{\tau}_{(n)}$. The parenthesized subscript n denotes that this is defined for n subscatterers. In this definition of the aggregate T matrix, the n subscatterers are regarded as one aggregate scatterer yielding a scattered field from the origin of a global coordinate system.

When n' subscatterers are added to the previous n subscatterer, resulting in $n' + 1$ scattering centers, the total field can be written as

$$\phi(r) = \Re g\psi^t(k_0, r_0) \cdot a + \psi^t(k_0, r_0) \cdot \overline{\tau}_{n(n+n')} \cdot a$$
$$+ \sum_{i=n+1}^{n+n'} \psi^t(k_0, r_i) \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \cdot a, \quad (2)$$

with $n' + 1$ unknowns, i.e., $\overline{\tau}_{n(n+n')}$ and $\overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}$, for $i = n + 1, \cdots, n + n'$. This is because the scattered field from the originally aggregated n subscatterers will be different due to multiple scattering: this new amplitude is denoted by $\overline{\tau}_{n(n+n')} \cdot a$. In addition, there are n' new scattering centers each of which has a scattered wave amplitude denoted by $\overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \cdot a$. Here, r_i is the field observation point with respect to the coordinates whose origin is at the center of the ith subscatterer. Alternatively, we can interpret $\overline{\tau}_{n(n+n')}$ as the aggregate T matrix for n subscatterers in the presence of $n + n'$ subscatterers, and $\overline{T}_{i(n+n')}$ is the $(n + n')$-scatterer T matrix for the ith scatterer (for detailed definition of notations, see [1]–[6]).

The aforementioned $n' + 1$ unknowns could be found as follows using the solution (1). First, we express the field from the n' subscatterers as incident field on the aggregated n subscatterers. To this end, we use the translation formula to express the scattered field from the n' subscatterers as standing waves about the global origin O, so that

$$\phi(r) = \Re g\psi^t(k_0, r_0) \cdot a + \Re g\psi^t(k_0, r_0)$$
$$\cdot \sum_{i=n+1}^{n+n'} \overline{\alpha}_{0i} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \cdot a$$
$$+ \psi^t(k_0, r_0) \cdot \overline{\tau}_{n(n+n')} \cdot a. \quad (3)$$

In the above, we have used the fact that [5, p. 464]

$$\psi^t(k_0, r_i) = \Re g\psi^t(k_0, r_0) \cdot \overline{\alpha}_{0i}, \quad |r_0| < d_{0i}, \quad (3a)$$

where d_{0i} is the distance between the origins of the ith coordinates and the global coordinates. The translation matrix $\overline{\alpha}_{0i}$ contains elements given by

$$[\overline{\alpha}_{0i}]_{nm} = H_{n-m}^{(1)}(k_0 d_{0i})e^{-i(n-m)\phi_{0i}}, \quad (3b)$$

where ϕ_{0i} is the angle the line d_{0i} makes with the x_0 axis.

The first and second terms of (3) can now be considered as the incident wave on the aggregate n subscatterers whose scattered field is given by the third term. Comparing (3) with (1), we notice that they have the same form, except that now, the strength of the incident field is changed. Hence from this comparison, we conclude that the amplitude of the scattered field from the aggregate n subscatterers must be just $\overline{\tau}_{(n)}$, the aggregate T matrix for n subscatterers, multiplied by the amplitude of the incident field, or

$$\overline{\tau}_{n(n+n')} = \overline{\tau}_{(n)} \cdot \left(\overline{I} + \sum_{i=n+1}^{n+n'} \overline{\alpha}_{0i} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}\right). \quad (4)$$

Next, we focus on the jth subscatterer of the group of n' subscatterers and express the fields from all the other subscatterers as incident fields on the jth subscatterer. By so doing, we can rewrite (2) as

$$\phi(r) = \Re g\psi^t(k_0, r_j) \cdot \overline{\beta}_{j0} \cdot a$$
$$+ \Re g\psi^t(k_0, r_j) \cdot \overline{\alpha}_{j0} \cdot \overline{\tau}_{n(n+n')} \cdot a$$
$$+ \Re g\psi^t(k_0, r_j) \cdot \sum_{\substack{i=n+1 \\ i \neq j}}^{n+n'} \overline{\alpha}_{ji} \cdot \overline{T}_{i(n+n')}\overline{\beta}_{i0} \cdot a$$
$$+ \psi^t(k_0, r_j) \cdot \overline{T}_{j(n+n')} \cdot \overline{\beta}_{j0} \cdot a, \quad (5)$$

where the summation is for $i = n + 1, \cdots, n + n'$, for $i \neq j$. In the above, we have used [5, p. 464]

$$\Re g\psi^t(k_0, r_0) = \Re g\psi^t(k_0, r_j) \cdot \overline{\beta}_{j0}, \quad (5a)$$

$$\psi^t(k_0, r_0) = \Re g\psi^t(k_0, r_j) \cdot \overline{\alpha}_{j0}, \quad |r_j| < d_{j0} \quad (5b)$$

$$\psi^t(k_0, r_i) = \Re g\psi^t(k_0, r_j) \cdot \overline{\alpha}_{ji}, \quad |r_i| < d_{ij} \quad (5c)$$

which are the representation of the addition theorem in matrix notation. The elements of $\overline{\alpha}_{j0}$ and $\overline{\alpha}_{ji}$ are similar to

that of (3b), but the elements of $\overline{\beta}_{j0}$ are Bessel functions instead.

Looking at (5), we note that the first three terms can be regarded as incident field on the jth subscatterer, and the last term is just the scattered field from the jth subscatterer. Therefore, the scattered field off the jth subscatterer must be related to the incident field via the isolated single-scatterer T matrix for the jth subscatterer. Consequently, we have

$$\overline{T}_{j(n+n')} \cdot \overline{\beta}_{j0} = \overline{T}_{j(1)} \cdot \left(\overline{\beta}_{j0} + \overline{\alpha}_{j0} \cdot \overline{\tau}_{n(n+n')} \right.$$
$$\left. + \sum_{\substack{i=n+1 \\ i \neq j}}^{n+n'} \overline{\alpha}_{ji} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \right),$$
$$j = n+1, \cdots, n+n', \quad (6)$$

where $\overline{T}_{j(t)}$ is the single-scatterer T matrix for the jth subscatterer.

Equation (4) can be used to substitute for $\overline{\tau}_{n(n+n')}$ in (6) to yield

$$\overline{T}_{j(n+n')} \cdot \overline{\beta}_{j0} = \overline{T}_{j(1)} \cdot \left[\overline{\beta}_{j0} + \overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)} \right.$$
$$\cdot \left(\overline{I} + \sum_{i=n+1}^{n+n'} \overline{\alpha}_{0i} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \right)$$
$$\left. + \sum_{\substack{i=n+1 \\ i \neq j}}^{n+n'} \overline{\alpha}_{ji} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \right],$$
$$j = n+1, \cdots, n+n'. \quad (7)$$

The above could be rearranged to yield

$$\overline{D}_j \cdot \overline{T}_{j(n+n')} \cdot \overline{\beta}_{j0}$$
$$= \overline{T}_{j(1)} \cdot \left[\overline{\beta}_{j0} + \overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)} + \sum_{\substack{i=n+1 \\ i \neq j}}^{n+n'} \right.$$
$$\left. \cdot (\overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)} \cdot \overline{\alpha}_{0i} + \overline{\alpha}_{ji}) \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} \right],$$
$$j = n+1, \cdots, n+n', \quad (8)$$

where $\overline{D}_j = [\overline{I} - \overline{T}_{j(1)} \cdot \overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)} \cdot \overline{\alpha}_{0j}]$. The above constitutes n' equations for the n' unknowns $\overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}$. It can be written more succinctly as

$$\sum_{i=n+1}^{n+n'} \overline{A}_{ji} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0} = -\overline{T}_{j(1)} \cdot [\overline{\beta}_{j0} + \overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)}],$$
$$j = n+1, \cdots, n+n', \quad (9)$$

where

$$\overline{A}_{ji} = \begin{cases} -\overline{D}_j, & i = j, \\ \overline{T}_{j(1)} \cdot \overline{\alpha}_{j0} \cdot \overline{\tau}_{(n)} \cdot \overline{\alpha}_{0i} + \overline{T}_{j(1)} \cdot \overline{\alpha}_{ji}, & i \neq j. \end{cases}$$
$$(9a)$$

After having solved (9) for $\overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}$, (4) could be used for find $\overline{\tau}_{n(n+n')}$.

When all the unknowns in the $n + n'$ subscatterer problem are solved for, then we can use

$$\psi^t(k_0, r_i) = \psi^t(k_0, r_0) \cdot \overline{\beta}_{0i}, \quad |r_0| > d_{0i}, \quad (10a)$$

in Equation (2), and finally rewrite (2) as

$$\phi(r) = \Re g \psi^t(k_0, r_0) \cdot a + \psi^t(k_0, r_0) \cdot \overline{\tau}_{(n+n')} \cdot a$$
$$(10b)$$

where $\overline{\tau}_{(n+n')}$ is the aggregate T matrix for $n + n'$ subscatterers defined as

$$\overline{\tau}_{(n+n')} = \overline{\tau}_{n(n+n')} + \sum_{i=n+1}^{n+n'} \overline{\beta}_{0i} \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}. \quad (11)$$

Equation (4) can be substituted into (11) to yield

$$\overline{\tau}_{(n+n')} = \overline{\tau}_{(n)} + \sum_{i=n+1}^{n+n'} (\overline{\beta}_{0i} + \overline{\tau}_{(n)} \cdot \overline{\alpha}_{0i}) \cdot \overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}.$$
$$(12)$$

Equations (9) and (12) permit us to find the $(n + n')$-subscatterer solution from the n-subscatterer solution. These equations could be used recursively to find the N subscatterer solution starting from a 0 subscatterer solution. When $n' = 1$, (9) and (12) reduce to the previously derived recursive algorithm [4], [5]. A backward recursion formula can be derived so that fields internal to the scatterer can be found [4], [7], [12].

III. COMPUTATIONAL COMPLEXITY

In the forward recursive algorithm, an n'-subscatterer problem is solved at each recursion so that n' linear algebraic equations are solved at each iteration as exemplified by (9). If P harmonics are used to expand the incident field in (1), and P harmonics are used to approximate the scattered field from the aggregated n subscatterers, then, $\overline{\tau}_{(n)}$ is a $P \times P$ matrix, and a is a $P \times 1$ vector. In (2), if M harmonics are used to approximate the scattered field from the ith subscatterer, where $i = n + 1$ to $n + n'$, then, $\overline{T}_{i(n+n')} \cdot \overline{\beta}_{i0}$ is an $M \times P$ matrix. In general, P harmonics are needed to keep the translation formulas accurate so that in (9), (9a), and (12), $\overline{\beta}_{j0}$ and $\overline{\alpha}_{j0}$ are $M \times P$, while $\overline{\alpha}_{0i}$ and $\overline{\beta}_{0i}$ are $P \times M$. Consequently, (9) constitutes solving a matrix equation with dimension $Mn' \times Mn'$. The inverse of this matrix equation by elimination techniques (e.g., Gauss Jordan) will be of $O(M^3 n'^3)$ complexity. Since there are P columns on the right hand side of (9), the cost of solving (9) for P right-hand sides will need an additional cost of $O(M^2 n'^2 P)$.

Since $\overline{\tau}_{(n)}$ is $P \times P$ where usually $P \gg M$, the cost of forming equation (9) is dominated by $O(Mn'P^2)$ which is the cost of forming $\overline{T}_{j(1)} \cdot \overline{\alpha}_{jo} \cdot \overline{\tau}_{(n)} \cdot \overline{\alpha}_{oi}$ in (10) and the right-hand side of (9). The cost of evaluating (11) is also of $O(Mn'P^2)$ (This analysis is similar to those presented in [3]–[6]). Consequently, the CPU time at each recursive step is given by

$$t \sim C_1(M^3 n'^3) + C_2(M^2 n'^2 P) + C_3(Mn' P^2) \quad (13)$$

where C_1, C_2 and C_3 are numbers of the same order that depend on implementations.

In two dimensions, if n' is the number of scatterers in the jth recursive step, then,

$$n' \simeq \frac{j}{L} (2\pi R \, \Delta D) \quad (14)$$

where L is the total number of recursive steps required to solve the scattering problem. (i.e., L is the number of shell layers into which the scatterer is divided), R is the maximum radius of the scatterer, Δ is the thickness of the shell, and D is the density of the subscatterers per unit cross-sectional area. Since $R = L \, \Delta$, we have

$$n' \simeq j 2\pi \, \Delta^2 D. \quad (15)$$

Consequently, the first term in (13) is approximately

$$t_{1j} \simeq C_1 (2\pi \, \Delta^2 D)^3 M^3 j^3. \quad (16)$$

After stepping through L steps, the total contribution to the CPU time from the first term is approximately[1]

$$T_1 = \sum_{j=1}^{L} t_{1j} \simeq \frac{C_1 (2\pi \, \Delta^2 D)^3}{4} M^3 L^4. \quad (17)$$

Since $L \, \Delta = R \cong \sqrt{N/\pi D}$, we have $L = \sqrt{N/\pi D \, \Delta^2}$. Substituting L into (17), we have

$$T_1 \simeq 2C_1 \pi \, \Delta^2 D M^3 N^2. \quad (18)$$

Using (15), the second term in (13) is

$$t_{2j} \simeq C_2 (2\pi \, \Delta^2 D)^2 M^2 j^2 P. \quad (19)$$

In two dimensions, $P \simeq (\pi \, d_j/2a)$ where d_j is the diameter of the object in the jth recursion and $2a$ is the diameter of the subscatterers.[2] Approximately, $2a \simeq \sqrt{1/D}$ and $d_j \simeq (j/L)(2R)$ where D and R is the density and the radius of the scatterer respectively. Therefore,

$$P \simeq \frac{j}{L} 2\pi R \sqrt{D} \simeq j 2 \sqrt{\pi} (\pi D \, \Delta^2)^{1/2}, \quad (20)$$

after using $R = \Delta L$. Consequently,

$$t_{2j} \simeq 8 \sqrt{\pi} C_2 (\pi \, \Delta^2 D)^{5/2} M^2 j^3. \quad (21)$$

After stepping through L steps, we have

$$T_2 = \sum_{j=1}^{L} t_{2j} \simeq C_2 2 \sqrt{\pi} (\pi \, \Delta^2 D)^{5/2} M^2 L^4$$

$$= C_2 2 \sqrt{\pi} (\pi \, \Delta^2 D)^{1/2} M^2 N^2. \quad (22)$$

A similar analysis shows that the contribution from the last term in (13) is

$$T_3 \simeq C_3 2\pi MN^2 \quad (23)$$

Therefore, the total CPU time in solving the problem is approximately

$$T \simeq [2C_1 M^3 \pi \, \Delta^2 D + C_2 2 \sqrt{\pi} (\pi \, \Delta^2 D)^{1/2} M^2$$

$$+ C_3 2\pi M] N^2. \quad (24)$$

If the shell thickness is such that $\pi \, \Delta^2 D$ is kept constant and the number of harmonics M for each subscatterer is constant,[3] then the CPU time grows as N^2. Note that the first term in (24), which comes from Gaussian elimination of (9), is proportional to $M^3 \, \Delta^2$. Hence it easily dominates the other terms if $M^3 \, \Delta^2$ is not small. However, the first two terms could be made small by making Δ small.[4] On the other hand, making Δ small gives rise to Gibbs phenomenon in the cylindrical harmonic expansion which is a Fourier series expansion, but this can be mitigated by windowing techniques [13], [14]. A similar analysis in three dimensions shows that the computer time grows as $N^{7/3}$.[5]

IV. NUMERICAL SIMULATIONS

We have applied this generalized algorithm to derive the scattering solution from an inhomogeneous cylinder. The inhomogeneous cylinder is first divided into N square subscatterers. The subscatterers are then replaced by circular subscatterers of the same cross-sectional area [15], so that their isolated single-scatterer T matrix, $\overline{T}_{i(1)}$, is readily obtained in closed form. As we shall show in the ensuing numerical experiment, the replacement of square subscatterers with circular subscatterers affects little the scattering solution.

Fig. 2 shows the scattered field of a square dielectric cylinder which is 0.9 free-space wavelengths square. The illuminating plane wave is H_z polarized. The scattering solution is first sought with the extended boundary condition (EBC) method [16]. Then the scatterer is replaced with 81 square cylindrical subscatterers whose isolated single-scatterer T matrix is also found with the EBC method. The linear algebraic equations associated with these 81 subscatterers are solved by a brute force method similar to that in [17]. Then the subscatterers are replaced with circular cylinders of the same cross-sectional area[6] so that the isolated-scatterer T matrix is easily sought. In this case, the circular cylinders are slightly overlapping.[7] The circular cylinders could be made smaller by the use

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[1] We have used the approximation $\sum_{j=1}^{L} j^3 \simeq L^4/4$.

[2] This follows from that for H_z wave scattering, each subscatter is predominantly dipolar and hence singular. An error analysis on the cylindrical harmonic expansion, which is a Fourier series expansion yields the above result. Also, the truncation of a Fourier series expansion of a function which is singular gives rise to Gibbs phenomenon [13], but this can be mitigated by windowing the Fourier series [13], [14].

[3] This is the case if each subscatterer is small and is predominantly dipole-like.

[4] The subscatterers are ordered by their centers. Hence when Δ is made small enough, an overlapping chain of subscatterers does not occur.

[5] The computation complexities for three dimensions reported in [3] and [4] are incorrect. In [3], it should be $N^{8/3}$, and in [4], it should be $N^{7/3}$.

[6] First used in [15], this is justified by the Maxwell-Garnett formula. The equal area assumption causes the circular and the square cylinders to have the same dipolar polarizability resulting in the same effective permittivity.

[7] This does not violate the addition theorem as the wave from a single subscatterer is coming out from the center of the subscatterer mathematically.

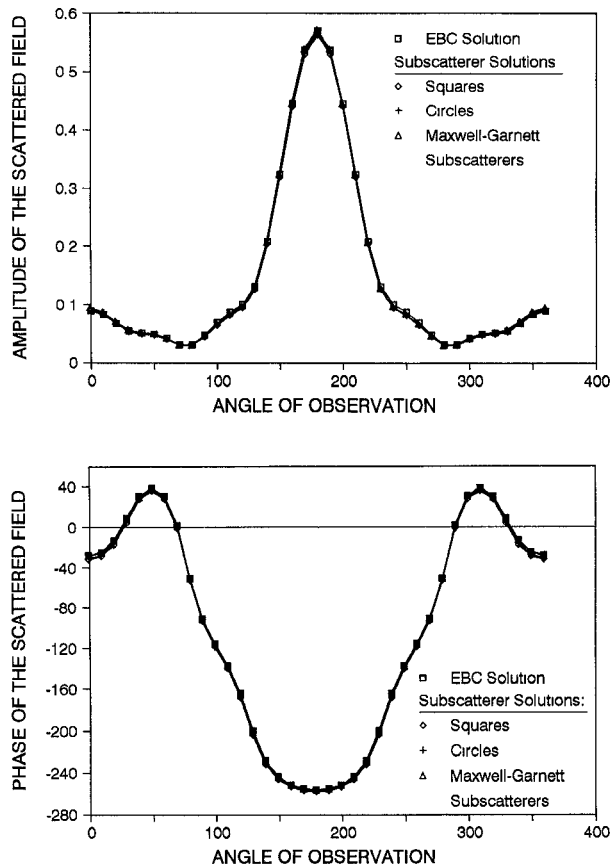Fig. 2. The scattering solution of a square cylindrical dielectric scatterer of dimension 0.9 free-space wavelengths square illuminated by an H_z polarized wave from the x axis at 300 MHz. The relative permittivity of the cylinder is 2, and the field is computed at 10 free-space wavelengths from the center of the scatterer. The square dots represent the solution calculated with the extended boundary condition (EBC) method using 21 harmonics. The diamonds represent the solution calculated with 81 square subscatterers. The crosses represent the solution when the square subscatterers are replaced with circular subscatterers of the same cross-sectional area. The triangles are when the circular subscatterers are reduced in size with the help of Maxwell–Garnett mixing formulas (see (25)) with a fractional cross-sectional area of 80%. All subscatterers have three harmonics of scattered field.

Fig. 3. Scattering of an H_z polarized incident plane wave using the method of moments, generalized recursive algorithm, the old recursive algorithm of [4], and analytic solution. (a) Compares the amplitude while (b) compares the phase. The frequency of the incident wave is 600 MHz. The dielectric cylinder has a diameter of 0.4 m with a relative permittivity of 2. The incident plane wave is from the x axis. The field is measured at 1 meter from the center of the cylinder. The circular cylinder is modeled with 137 subscatterers. Each subscatterer has three harmonics of scattered field, and 40 terms are used to keep the translation formulas accurate.

of the Maxwell–Garnett mixing formula [18] in two dimensions which is

$$\epsilon_s = \epsilon_0 \frac{1 + \dfrac{1}{f}\dfrac{\epsilon_e - \epsilon_0}{\epsilon_e + \epsilon_0}}{1 - \dfrac{1}{f}\dfrac{\epsilon_e - \epsilon_0}{\epsilon_e + \epsilon_0}}, \qquad (25)$$

where ϵ_e is the desired permittivity of the mixture, f is the fractional cross-sectional area of the circular cylinders, and ϵ_s is the permittivity of the circular cylinders that will yield the desired mixture permittivity. The solution with Maxwell–Garnett subscatterers has a fractional area of 80% which shrinks them enough so that they do not overlap. The solutions for all four methods are seen to be in good agreement. The agreement could be attributed to the fact that when the scattering properties of the subscatterers are replaced by the isolated-scatterer T matrix, the
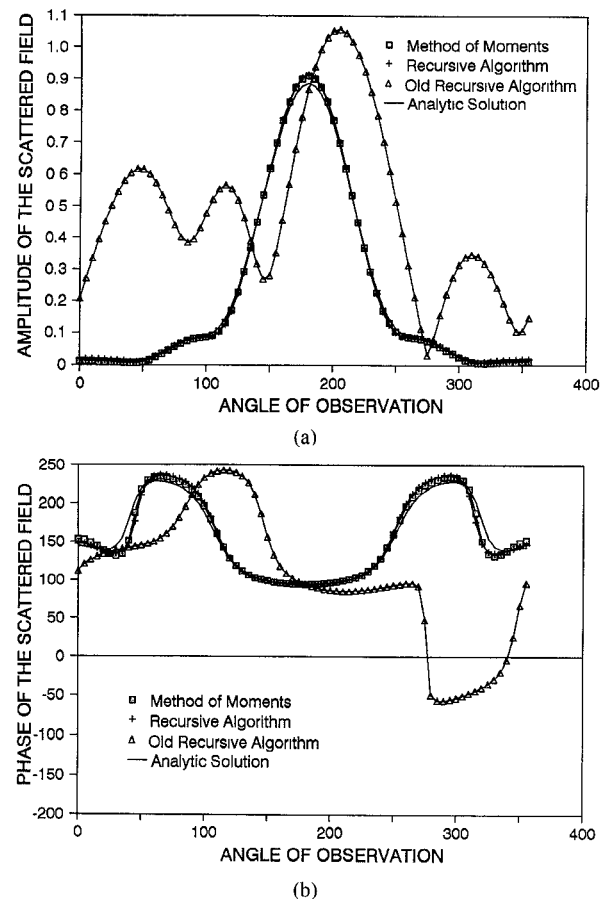
subscatterers are behaving like points subscatterers. Hence, the shape of the subscatterers is immaterial as long as they have the same scattering strength.

Fig. 3 shows the H_z scattering solution from a one wavelength circular dielectric scatterer comparing the method of moments (MOM) solution [15], [18], the analytic solution and the generalized recursive algorithm and the old recursive algorithm [4]. The old recursive algorithm is seen to fail in this case. It is seen that the recursive algorithm agrees better with the MOM solution than with the closed-form solution. We trace the disagreement with closed-form solution to modeling imperfections since the circular scatterer is replaced with discrete subscatterers. The outer boundary of the resultant scatterer is no longer perfectly circular, but both the MOM solution and the recursive algorithm solution are modeling this imperfectly circular scatterer, which is different from that modeled by the analytic solution.

Fig. 4(a) shows the RCS of an inhomogeneous, circular, dielectric cylinder computed with the generalized recursive algorithm and the method of moments for H_z
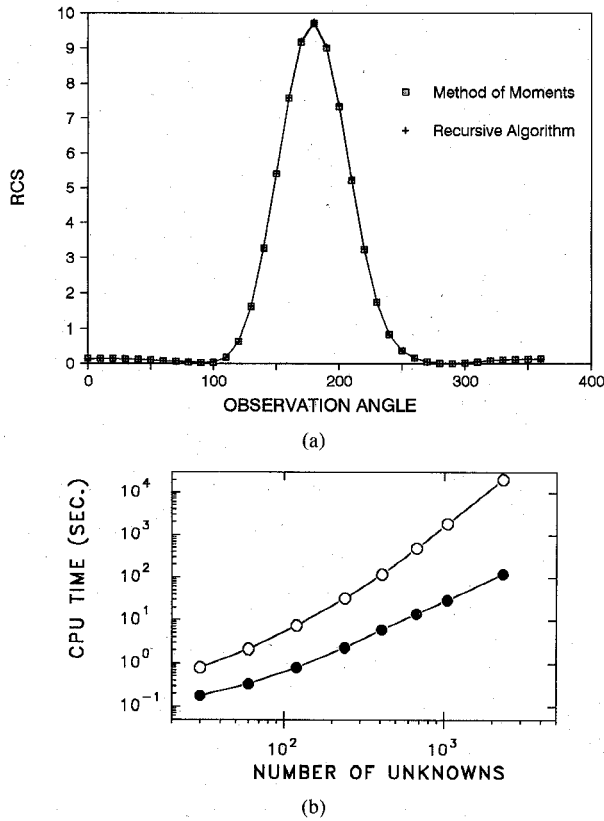
(a)

(b)

Fig. 4. (a) The RCS of an inhomogeneous dielectric cylinder computed using the generalized recursive algorithm and the method of moments. The frequency of the H_z polarized incident wave is 100 MHz. The inhomogeneous dielectric cylinder has a diameter of 3 m with an inhomogeneous dielectric distribution given by (26). The incident plane wave is from the x axis. The cylinder is modeled with 121 subscatterers. Each subscatterer has three harmonics of scattered field, and 70 terms are used to keep the translation formulas accurate. (b) Comparison of the CPU time versus the number of unknowns (MN) in the problems. The white circles are for the method of moments while the black circles are for the generalized recursive algorithm.

waves. The inhomogeneity is described by the equation

$$\epsilon_r = 1 + \frac{1 + \sin \phi}{2} \sin (\rho\pi /a), \qquad (26)$$

where a is the radius of the cylinder. For such an inhomogeneous cylinder, the scattering solution does not have a closed form. Hence, the method of moments is our only basis for comparison. Excellent agreement is observed between the generalized recursive algorithm and the method of moments, because both the solutions are modeled similarly.

Fig. 4(b) shows the growth of the CPU time on the CRAY-2 vesus the number of unknowns (MN) in the problem. Because of its reduced computational complexity, the CPU time for the recursive algorithm (black circles) grows slower than that for the method of moments (white circles). We expect this gap to widen as the number of unknowns gets large.[8]

[8]The overhead associated with Bessel function computation is small in this algorithm if they are calculated with recurrence relations as indicated by a table in [6].
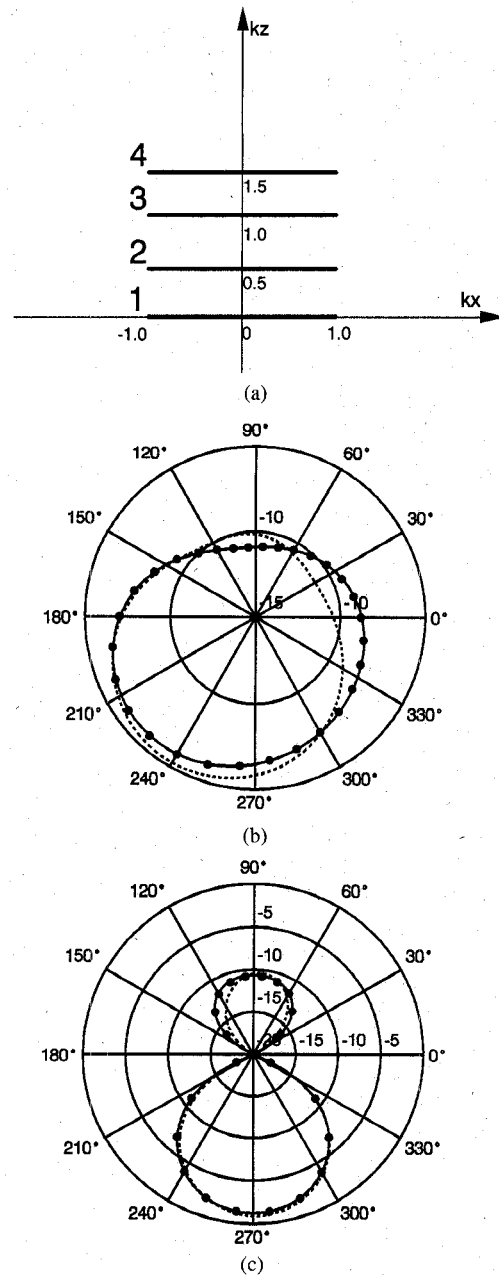

(a)

(b)

(c)

Fig. 5. Comparison of the RCS of an array of strips computed using the generalized recursive algorithm (solid circles), method of moments (solid lines) and the recursive algorithm presented in [3] (dotted lines). The incident wave makes an angle of 45 degrees with the x axis, and the plane of incidence is the xz plane. (a) Cross-section view of the geometry drawn to scale. (b) RCS for E_z polarized wave. (c) RCS for H_z polarized wave.

Fig. 5(a) illustrates the cross-sectional view of a two-dimensional, perfectly conducting strip geometry. Fig. 5(b) and (c) show the normalized radar cross sections (RCS's) of this geometry for E_z and H_z polarized waves, respectively. On both graphs, we present three sets of data:

1) The solid curves are the RCS's computed using the MOM.

2) The dashed curves are the RCS's computed using the recursive algorithm of [3], i.e., by adding the scatterers one at a time.

3) The solid circles are the RCS's computed using the generalized recursive algorithm of this paper by adding two strips at a time.

In both cases 2 and 3, each strip is treated as a subscatterer whose T matrix is known. The agreement between the algorithm of this paper and the MOM is seen to be excellent. On the other hand, when the strips are added to the geometry one by one using the algorithm of [3], the addition theorems are violated, and the RCS cannot be computed accurately. The T matrices for individual strips are calculated using a method presented in [20].

V. CONCLUSION

A generalization of the recursive algorithm previously described is reported here. This new recursive algorithm avoids the violation of the addition theorem. This violation has caused previously reported algorithms not to work well for H_z polarized wave scattering. The generalized recursive algorithm, however, remedies the problem of violating the addition theorem, and is demonstrated to work for H_z polarized wave scattering. Furthermore, it has a reduced computational complexity of $O(N^2)$ compared to MOM which has an $O(N^3)$ complexity. Unlike the conjugate gradient method which furnishes a solution valid for one incident wave only, this algorithm provides a solution valid for all angles of incidence. Moreover, the radiation condition at infinity is automatically satisfied by such a solution.

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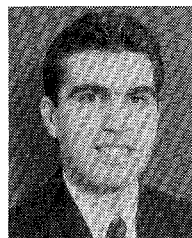
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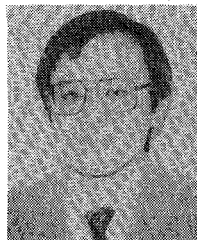
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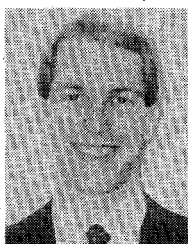
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